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Hahn–Jordan Decomposition for Gleason Measures

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The possibility of the Hahn-Jordan decomposition for **n**-finite signed measures, where **n** is a cardinal, defined on a quantum logic of all closed subspaces of a Hilbert space whose dimension is a nonmeasurable cardinal $\neq 2$, is investigated.

1. INTRODUCTION AND PRELIMINARIES

A signed measure on a quantum logic $\mathscr{L}(H)$ of all closed subspaces of a Hilbert space H (not necessarily separable) over the field C of real or complex numbers is a map m: $\mathscr{L}(H) \to [-\infty, \infty]$ such that: (1) $m(0)=0$; (2) m is σ-additive on all sequences of mutually orthogonal subspaces of $\mathscr{L}(H)$; (3) it attains at most one of the possible values $\pm\infty$. A positive signed measure is said to be a measure.

The well-known theorem of Gleason (1957) says that any finite measure m on $\mathscr{L}(H)$ of a separable Hilbert space H, $\dim H \neq 2$, is in one-to-one correspondence with a positive Hermitian operator T on H with a finite trace via

$$m(H) = \mathrm{tr}(TM), \qquad M \in \mathscr{L}(H) \tag{1}$$

(We identify a subpace M with its orthoprojector P^M on it.)

We say that $M \in \mathscr{L}(H)$ is positive [negative] with respect to m if, for any $N \subset M$, $N \in \mathscr{L}(H)$, $m(N) \geq 0$ [$m(N) \leq 0$]. A Jordan decomposition of a signed measure m is a pair (m_1, m_2) of measures on $\mathscr{L}(H)$ such that $m = m_1 - m_2$. A Hahn decomposition corresponding to m is an element $M \in \mathscr{L}(H)$ such that M is positive and $M^\perp$ is negative with respect to m.

A signed measure m is said to be: (1) bounded if $\sup\{|m(P)|: P \in \mathscr{L}(H)\} < \infty$; (2) semibounded if $\inf\{m(P_x): x \in H\} > -\infty$ (P_x is the one-dimensional subspace of H spanned over $x \neq 0$); (3) **n**-finite if there is a

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system of mutually orthogonal subspaces $\{M_a: a \in A\}$ such that $H = \bigoplus_{a \in A} M_a$, $|m(M_a)| < \infty$ for each $a \in A$, and card $A = \mathbf{n}$. In particular, if $\mathbf{n} = \aleph_0$ (the cardinal of all integers), we say that m is σ-finite. We recall that by $\bigoplus_{i \in I} P_i$ we mean the join of mutually orthogonal subspaces $\{P_i: i \in I\}$.

Sherstnev (1974) proved that if m is a finite, bounded, signed measure on $\mathscr{L}(H)$ of a separable Hilbert space H, dim $H \neq 2$, then m is expressible via (1), and, moreover, m admits Hahn and Jordan decompositions. This result has been generalized by Drisch (1979) to all bounded, signed measures on $\mathscr{L}(H)$ of a Hilbert space H whose dimension is a nonmeasurable cardinal $\neq 2$. We recall, according to Ulam (1930), that a cardinal I is said to be nonmeasurable if there is no nontrivial, positive, finite measure ν on the power set 2^A of a set A, card $A = I$, such that $\nu(\{a\}) = 0$ for any $a \in A$. For example, any finite cardinals, $\aleph_0$, $\mathbf{c}$ (cardinal of reals) (under the continuum hypothesis) are nonmeasurable.

Dvurečenskij (1987) has proved that, for any finite, signed measure m on a quantum logic $\mathscr{L}(H)$ of a Hilbert space H whose dimension is a nonmeasurable cardinal $\neq 2$, (1) holds iff m is bounded from below on all one-dimensional subspaces of H.

Without loss of generality, we shall suppose in the following that any signed measure attains from values $\pm\infty$ only $+\infty$. By a Gleason measure we shall mean any semibounded, signed measure.

First we remark that, for any finite-dimensional Hilbert space H, dim $H \geq 2$, there is an unbounded finite measure (Sherstnev, 1974; Dvurečenskij, 1987); consequently, there are signed measures that admit no Jordan decompositions.

A bilinear form is a function $t: D(t) \times D(t) \to C$, where $D(t)$ is submanifold in H, called the domain of definition of t, such that t is linear in the first argument and antilinear in the second one, and $t(\alpha x, y) = \alpha t(x, y)$ for all $x, y \in D(t)$, $\alpha \in C$. A bilinear form is: (1) symmetric if $t(x, y) = \overline{t(y, x)}$ for all $x, y \in D(t)$; (2) positive if $t(x, x) \geq 0$ for all $x \in D(t)$; (3) semibounded if there is a constant $K \geq 0$ such that $t(x, x) \geq -K$ for all $x \in D(t)$. If $P \subset D(t)$, then by $t \circ P$ we mean a bilinear form defined by $t \circ P(x, y) := t(Px, Py)$, $x, y \in H$. If $t \circ P$ is induced by a trace operator, we say $t \circ P \in \mathrm{Tr}(H)$, where $\mathrm{Tr}(H)$ is the class of all trace operators in H.

2. PROPERTIES OF SIGNED MEASURES

The positive and negative variations m^+ and m^- of a signed measure m are defined by

$$m^+(M) = \sup\{m(N): N \subset M\}$$

and

$$m^-(M) = \qquad -\inf\{m(N)\colon N \subset M\}$$

The total variation of m is the map $|m| := m^+ + m^-$.

Lemma 1. If (m_1, m_2) is a Jordan decomposition of a signed measure m, then $m_1 \geq m^+$ and $m_2 \geq m^-$.

Proof. We have

$$\begin{aligned} m^+(M) &= \sup\{m(N)\colon N \subset M\} = \sup\{m_1(N) - m_2(N)\colon N \subset M\} \\ &\leq \sup\{m_1(N)\colon N \subset M\} = m_1(M). \end{aligned}$$

Similarly, one can prove the second assertion. ■

Hence, in order to construct a Jordan decomposition for a finite m, it is enough to find a finite measure $m_1 \geq m^+$. If we define $m_2 = m_1 - m$, then m_2 is a measure, too, and (m_1, m_2) is the desired Jordan decomposition. Moreover, if (m_1, m_2) is any Jordan decomposition for m, and m_0 is a finite measure, then $(m_1 + m_0, m_2 + m_0)$ is a Jordan decomposition for m, too.

Remark 1. Theorem 7 in Dvurečenskij (1987) says that if a finite signed measure m admits a Jordan decomposition, then the Gleason theorem holds for m, and, consequently, there exists a Hahn decomposition corresponding to m. The first part of Proposition 6 in Dvurečenskij (1987) shows that on $\mathscr{L}(H)$, $\dim H = 2$, there are signed measures that attain no Jordan decompositions; however, any one-dimensional subspace of H forms a Hahn decomposition corresponding to m.

Therefore, it would be interesting to examine whether the existence of a Hahn decomposition implies the existence of a Jordan one when $\dim H \geq 3$.

Lemma 2. Let m be a signed measure on $\mathscr{L}(H)$ of an arbitrary Hilbert space H. Let $M \in \mathscr{L}(H)$ be such $0 < m(M) \leq \infty$. Then there is N, $N \subset M$, that is positive with respect to m and $m(N) > 0$.

Proof. Either M itself is positive with respect to m or it contains subspaces of negative measure. In the latter case let n_1 be the smallest positive integer such that there is a subspace $M_1 \subset M$ with $m(M_1) < -1/n_1$.

Proceeding inductively, let n_k be the smallest positive integer for which there is M_k such that $M_k \subset M \cap (\bigoplus_{i=1}^{k-1} M_i)^\perp$ and $m(M_k) < -1/n_k$. If we put $N = M \cap (\bigoplus_{i=1}^{\infty} M_i)^\perp$, then $M = N \oplus \bigoplus_{i=1}^{\infty} M_i$. Hence,

$$m(M) = m(N) + \sum_{i=1}^{\infty} m(M_i)$$

The series on the right-hand absolutely converges since $m(\bigoplus_{i=1}^{\infty} M_i)=\sum_{i=1}^{\infty} m(M_i)$ is finite. Thus, $\sum_i 1/n_i$ converges, and we have $n_i \to \infty$. It is clear that $m(N)>0$.

To show that N is positive with respect to m, let $\varepsilon>0$ be given. Since $n_i \to \infty$, we may choose i so large that $1/(n_i-1)<\varepsilon$. Since

$$N \subset M \cap \left(\bigoplus_{j=1}^{i} M_j\right)^{\perp} \tag{2}$$

N can contain no subspaces with measure less than $-1/(n_i-1)$ which is greater than $-\varepsilon$. Hence, N contains no subspaces less than $-\varepsilon$. Since ε is an arbitrary positive number, it follows that N can contain no subspaces of negative measure, and, therefore, must be positive with respect to m. ∎

3. FINITE-DIMENSIONAL HILBERT SPACE

First we investigate the possibility of Jordan and Hahn decompositions for Gleason measures on a quantum logic of a finite-dimensional Hilbert space.

Theorem 3. Let m be a Gleason measure on $\mathscr{L}(H)$ of a finite-dimensional Hilbert space H for which there is a three-dimensional Q with $m(Q)<\infty$. Then there exist Jordan and Hahn decompositions corresponding to m.

Proof. First we suppose that $m(H)$ is finite. Due to Dvurečenskij (1987), m is representable via (1). Let T^+ and T^- be the positive and negative parts of the Hermitian operator T. Then (m_1, m_2), where $m_1(M):=\operatorname{tr}(T^+M)$, $m_2(M):=\operatorname{tr}(T^-M)$, $M\in\mathscr{L}(H)$, is a Jordan decomposition for m. Now let H^+ and H^- be subspaces generated by proper vectors of T^+ and T^-. Then they are positive and negative with respect to m, and $H^+\oplus H^-=H$.

Now we suppose $m(H)=\infty$. Proposition 2 and Theorem 3 in Dvurečenskij (1987) imply that there is a subspace P, $3\le \dim P\le n-1$, such that $m(M)<\infty$ if $M\subset P$. Then $m_0:=m|\mathscr{L}(P)$ is a bounded, signed measure and m has the form

$$m(M)=\begin{cases} m_0(M) & \text{if} \quad M\subset P \\ \infty & \text{otherwise} \end{cases} \tag{3}$$

The first part of the present theorem shows that for m_0 there is a Jordan decomposition (m_0^+, m_0^-), and, moreover, m_0^+ and m_0^- are representable via (1) on $\mathscr{L}(P)$ through positive Hermitian operators T_0^+ and T_0^- in a Hilbert space P. Let $\bar{T}_0^-$ be the extension to whole H defining $\bar{T}_0^- x = 0$ for any $x \perp P$. Define two measures m_1 and m_2 on $\mathscr{L}(H)$ via

$$m_1(M) = \begin{cases} m_0^+(M) & \text{if} \quad M \subset P \\ \infty & \text{otherwise} \end{cases}$$

$$m_2(M) = \operatorname{tr}(\bar{T}_0^- M), \qquad M \in \mathscr{L}(H)$$

The simple verification shows that (m_1, m_2) is a Jordan decomposition for m.

Let now $P = P_+ \oplus P_-$, where P_+ and P_- are positive and negative, respectively, with respect to m_0. Define $M_- = P_-$, $M_+ = M_-^\perp$. The M_- is negative with respect to m. Let now $M \subset M_+$. Then either $M \not\subset P$ or $M \subset P$. In the first case $m(M) = \infty$, and in the latter one we can show that $M \subset P_+$, that is, $m(M) \geq 0$, and this finishes the proof. ■

4. DECOMPOSITIONS FOR UNBOUNDED GLEASON MEASURES

Dvurečenskij (1987) proved the following theorem, which is a generalization of Gleason's theorem to unbounded, semibounded signed measures.

Theorem 4 (A. R. Gleason). Let $\mathbf{n}$ be a cardinal and let m be an $\mathbf{n}$-finite Gleason measure, $m(H) = \infty$, on a quantum logic $\mathscr{L}(H)$ of a Hilbert space H whose dimension is a nonmeasurable cardinal $\neq 2$. Then there is a unique symmetric semibounded, bilinear form t with a dense domain such that

$$m(P) = \begin{cases} \operatorname{tr} t \circ P & \text{if} \quad m(P) < \infty \\ \infty & \text{otherwise} \end{cases} \tag{4}$$

Moreover, if $m(\bigoplus_{a \in A} M_a) < \infty$, then

$$m\left(\bigoplus_{a \in A} M_a\right) = \sum_{a \in A} m(M_a) \tag{5}$$

In the following we shall investigate the possibility of Hahn-Jordan decompositions for Gleason measures with $m(H) = \infty$.

Let t be a symmetric bilinear form with a domain $D(t)$. We say that a number λ is a lower proper value of t on a subspace M of H if

$$\lambda = \inf\{t(x, x) \colon \|x\| = 1, x \in M \cap D(t)\}$$

The unit vector $x \in D(t) \cap M$ is a proper vector of t on M corresponding to the lower proper value λ if

$$\lambda = t(x, x)$$

Lemma 5. Let t be a symmetric, semibounded, bilinear form on $D(t)$. Let there exist $x_0 \in D(t)$, $\|x\| = 1$, such that $t(x_0, x_0) = \inf\{t(x, x): \|x\| = 1, x \in D(t)\}$. If $y \perp x_0$ and $y \in D(t)$, then $t(x_0, y) = 0$.

Proof. Let $K = -t(x_0, x_0)$. From the Schwarz inequality we have

$$\begin{aligned} 0 &\leq |t(x_0, y) + K(x_0, y)| = |t(x_0, y)| \\ &\leq [(t(x_0, x_0) + K][t(y, y) + K\|y\|^2)]^{1/2} \\ &= 0 \quad \blacksquare \end{aligned}$$

Lemma 6. Let the conditions of Theorem 4 hold and let t be a bilinear form from (4). Let $\lambda < 0$ be the lower proper value of t on m. Denote $M(\lambda) = \{x \in M \cap D(t): t(x, x) = \lambda \|x\|^2\}$. Then $M(\lambda)$ is a finite-dimensional subspace of H.

Proof. It is evident $M(\lambda) \neq \emptyset$. We claim to show that if $x_1, x_2 \in M(\lambda)$, then $x_1 + x_2 \in M(\lambda)$, then $x_1 + x_2 \in M(\lambda)$. Suppose x_1 and x_2 are linearly independent vectors. Since $x_1 + x_2 \in D(t)$, there is a Hermitian operator T on $N = P_{x_1} \vee P_{x_2}$ such that $t(z, z) = (Tz, z)$, $z \in N$. We show that $(Tz, z) \leq 0$ for any $z \in N$. If not, then there is $z_0 \neq 0$ with $(Tz_0, z_0) > 0$. Hence, there are two orthogonal unit vectors z_1 and z_2 and two numbers $\mu > 0$ and $\lambda_0 < 0$ such that $Tz_1 = \mu z_1$, $Tz_2 = \lambda_0 z_2$. It is evident that $\lambda_0 = \inf\{t(z, z): z \in N\} = \lambda$. Let $x_1 = az_1 + bz_2$, where $|a|^2 + |b|^2 = \|x_1\|^2$. Then

$$t(x_1, x_1) = (Tx_1, x_1) = \mu|a|^2 + \lambda_0|b|^2 = \lambda_0\|x_1\|^2$$

Hence, $|b|^2 = \|x_1\|^2$, $a = 0$, which gives $x_1 = bz_2$. Analogously, we obtain that $x_2 = cz_2$ for some scalar c. In other words, x_1 and x_2 are linearly dependent, which is a contradiction. Therefore, $(Tz, z) \leq 0$ for any $z \in N$, so that $Tx_i = \lambda x_i$, $i = 1, 2$, and, moreover, $x_1 + x_2 \in M(\lambda)$.

Now suppose that in $M(\lambda)$ there are infinitely many linearly independent vectors, $\{e_n\}_{n=1}^{\infty}$ say. Using the Gram-Schmidt orthogonalization process to $\{e_n\}_{n=1}^{\infty}$, we may find a sequence of orthonormal vectors $\{f_n\}_{n=1}^{\infty}$ in $M(\lambda)$. Then

$$m\left(\bigoplus_{n=1}^{\infty} P_{f_n}\right) = \sum_{n=1}^{\infty} m(P_{f_n}) = \sum_{n=1}^{\infty} t(f_n, f_n) = -\infty$$

Hence, $M(\lambda)$ is a finite-dimensional subspace of H. $\blacksquare$

We need the following important assumption:

Hypothesis. For any σ-finite Gleason measure m on $\mathscr{L}(H)$ of a separable Hilbert space H, there is a unit vector x such that

$$m(P_x) = \inf\{m(P_y): \|y\| = 1, y \in H\} \tag{6}$$

If m is a finite Gleason measure, then the hypothesis is true (even if H has dimension of a nonmeasurable cardinal $\neq 2$). The same is true if a bilinear form t from Theorem 4 is closed, that is, if $x_n \to x$, $\{x_n\} \subset D(t)$, $t(x_n - x_m, x_n - x_m) \to 0$ imply $x \in D(t)$ and $t(x_n - x, x_n - x) \to 0$, moreover, Jordan and Hahn decompositions are obtainable. Indeed, there is a unique self-adjoint operator $T = T_+ - T_-$, where $D(t) = D(T_+^{1/2})$, and due to the spectral theorem, T_- is bounded, such that $t(x, x) = (T_+^{1/2}x, T_+^{1/2}x) - (T_-x, x)$, $x \in D(t)$. Moreover, $T_- \in \mathrm{Tr}(H)$ and $\|T_-\| = -\inf\{t(x, x): \|x\| = 1, x \in D(t)\}$. Let E be a spectral measure corresponding to T. Then $M^+ := E([0, \infty))$ and $M^- := E((-\infty, 0))$ are, respectively, positive and negative with respect to m. A bilinear form $t_+(x, x) := (T_+^{1/2}x, T_+^{1/2}x)$ with $D(t_+) = D(T_+^{1/2})$ determines a σ-finite measure m_+ on $\mathscr{L}(H)$ via (4) and $t_-(x, x) := (T_-x, x)$ determines a finite measure m_-. Moreover, $m = m_+ - m_-$.

I do not know if the above hypothesis is true in general.

Theorem 7. (Hahn–Jordan decomposition). Let $\mathbf{n}$ be a cardinal and let m be an $\mathbf{n}$-finite Gleason measure on $\mathscr{L}(H)$ of a Hilbert space H whose dimension is a nonmeasurable cardinal $\neq 2$ and let $m(H) = \infty$. Under the hypothesis, m admits Hahn and Jordan decompositions.

Proof. Let t be a symmetric bilinear form determined by m whose existence follows from Theorem 4. Put $\lambda_1 = \inf\{t(x, x): \|x\| = 1, x \in D(t)\}$. Hence, there is a sequence of unit vectors $\{x_n\}_{n=1}^{\infty}$ belonging to $D(t)$ such that $\lambda_1 = \lim_n t(x_n, x_n)$. Put $H_1 = \bigvee_{n=1}^{\infty} P_{x_n}$ and $m_1 := m|\mathscr{L}(H_1)$. Then m_1 is a σ-finite Gleason measure on $\mathscr{L}(H_1)$ and, due to the hypothesis, there is a unit vector $e_1 \in D(t)$ such that $\lambda_1 = t(e_1, e_1)$.

Let $\lambda_2 = \inf\{t(x, x): x \in D(t), \|x\| = 1, x \in M_1\}$, where $M_1 = P_{e_1}^{\perp}$. Then either $\lambda_2 \geq 0$ or $\lambda_2 < 0$. In the first case P_{e_1} is negative and M_1 positive with respect to m. In the second case, as above, we find $e_2 \in D(t) \cap M_1$, $\|e_2\| = 1$, such that $\lambda_2 = t(e_2, e_2)$.

Proceeding inductively, suppose that we have constructed orthonormal vectors $e_i \in D(t) \cap M_i$, $i = 1, \ldots, n$, and we find negative numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 0$ such that $\lambda_i = \inf\{t(x, x): x \in D(t) \cap M_i, \|x\| = 1\}$ and $\lambda_i = t(e_i, e_i)$, where $M_i = (\bigoplus_{j=1}^{i-1} P_{e_j})^{\perp}$.

There are two possible cases: (1) After a finite number of steps we obtain a subspace $M_{n_0}^{\perp}$ in which $t(x, x) \geq 0$, $x \in D(t) \cap M_{n_0}^{\perp}$; (2) $t(x, x) < 0$ on $D(t) \cap M_n^{\perp}$ for any integer $n \geq 1$.

The first case gives us $t(x, x) < 0$ for any $x \in D(t) \cap M_{n_0}$. In fact, let $x = \sum_{i=1}^{n_0} a_i e_i \in M_{n_0} \cap D(t)$. Then $t(x, x) = \sum_{i,j}^{n_0} a_i \bar{a}_j t(e_i, e_j)$ and, following Lemma 5, $t(e_i, e_j) = 0$ whenever $i \neq j$. Hence, $t(x, x) = \sum_{i=1}^{n_0} |a_i|^2 \lambda_i < 0$, which entails that M_{n_0} is negative with respect to m.

In the latter case we claim to show that if $M = (\bigoplus_{n=1}^{\infty} P_{e_n})^{\perp} = \bigwedge_{n=1}^{\infty} M_n^{\perp}$, then, for any $x \in M \cap D(t)$, $\|x\| = 1$, $t(x, x) \geq 0$. If not, then $x \in M_n^{\perp}$ for any $n \geq 1$. Thus, $\lambda_n \leq t(x, x) < 0$. On the other hand, $m(\bigoplus_{n=1}^{\infty} P_{e_n}) = \sum_{n=1}^{\infty} \lambda_n < 0$. The convergent series implies $\lambda_n \to 0$, that is, $t(x, x) = 0$.

Moreover, $t(x, x) \leq 0$ on $M^{\perp}$. In fact, since $m(M^{\perp}) < \infty$, there is a unique Hermitian operator $T \in \mathrm{Tr}(M^{\perp})$ such that $t(x, x) = (Tx, x)$, $x \in M^{\perp}$. Representing $x = \sum_{n=1}^{\infty} (x, e_n) e_n$ and using Lemma 5, we have $t(x, x) \leq 0$.

We assert that M (in case 1 we put $M = M_{n_0}^{\perp}$) is positive with respect to m. If not, there is $N \subset M$ such that $m(N) < 0$. Hence, there is a Hermitian operator $T_N \in \mathrm{Tr}(N)$ such that $m(Q) = \mathrm{tr}(T_N Q)$, $Q \subset N$. Therefore $m_N := m|\mathscr{L}(N)$ is totally additive and, consequently, $m(N) = \sum_i m(P_{x_i}) \geq 0$, where $\{x_i\}$ is an orthonormal basis in N, and this gives a contradiction.

Put $M_+ = M$, $M_- = M^{\perp}$ and define two positive symmetric bilinear forms t_+, t_- via

$$D(t_+) = D(t) = D(t_-)$$

$$t_+(x, x) = t(M_+ x, M_+ x)$$

$$t_-(x, x) = -t(M_- x, M_- x)$$

Since $M_- \subset D(t)$, t_+ and t_- are defined well. Choose $x \in D(t)$ and calculate

$$t(x, x) = t(M_+ x, M_+ x) + t(M_- x, M_- x) + 2 \operatorname{Re} t(M_- x, M_+ x)$$

Since $m(M_- \oplus P_y) < \infty$, where $y = M_+ x$, we have that t is continuous on $M_- \oplus P_y$. Therefore,

$$t(M_- x, M_+ x) = \sum_{n=1}^{\infty} (M_- x\, e_n) t(e_n, M_+ x)$$

Lemma 5 implies $t(M_- x, M_+ x) = 0 = t(M_+ x, M_- x)$. Hence, we obtain

$$t(x, x) = t_+(x, x) - t_-(x, x), \qquad x \in D(t)$$

Now we construct a Jordan decomposition. Let T be the above positive Hermitian operator belonging to $\mathrm{Tr}(M_-)$. It may be extended to a positive Hermitian operator on whole H with the same trace, denoted as T_-. Define a finite measure $m_2(P) = \mathrm{tr}(T_- P)$, $P \in \mathscr{L}(H)$. Then m_1 determined by

$$m_1(M) = m(M) + m_2(M)$$

is an **n**-finite Gleason measure on $\mathscr{L}(H)$. Moreover, $|m_1(P_x)| < \infty$ iff $m(P_x) < \infty$. Hence, due to Theorem 4, m_1 is determined by a symmetric bilinear form equal to t_+, so that m_1 is a positive n-finite measure, and $m = m_1 - m_2$, which finishes the proof. ■

Corollary 8. Under the conditions of Theorem 7, m is totally additive on $\mathscr{L}(H)$, and $m(P) < \infty$ iff $t \circ P \in \mathrm{Tr}(H)$.

Proof. From Theorem 7 we have that $m = m_1 - m_2$. Here m_1 is n-finite and m_2 is a finite measure. Due to Dvurečenskij (1986), m_1 is totally additive and $m_1(P) < \infty$ iff $t_1 \circ P \in \mathrm{Tr}(H)$, where t_1 is a positive symmetric bilinear form corresponding to m_1 via (4). ■

Finally we show that in some particular cases we may find Hahn and Jordan decompositions for σ-finite Gleason measures without calling upon the hypothesis given earlier.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in H, $\dim H = \aleph_0$, which is a part of a Hamel basis $\{g_t : t \in T\}$. Fix a unit vector g_{t_0} from the Hamel basis that does not belong to the orthonormal one. Define an operator B via $B(\sum_{i \in T_0} \alpha_i g_i) = a_{t_0} g_{t_0}$, where T_0 is any finite subset of T containing t_0, and put $t(x, x) = (Bx, Bx)$, $x \in H$. Then t is a positive symmetric bilinear form. Choose a positive operator $T = \sum_{n=1}^{\infty} \lambda_n P_{e_n}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\lambda_1 > 1$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Define a semibounded, symmetric, bilinear form p with $D(p) = H$ via

$$p(f, g) = t(f, g) + \sum_{n=1}^{\infty} (1 - \lambda_n)(f, e_n)(e_n, g)$$

This form is not closed and it determines via (4) a σ-finite Gleason measure m on $\mathscr{L}(H)$ such that $m(M) < \infty$ iff $\dim M < \infty$. In this case we may find finitely many vectors $e_1, \ldots, e_{n_0}$ such that $1 - \lambda_i = \inf\{p(f, f) : f \in M_i, \|f\| = 1\}$, $i = 1, \ldots, n_0$, where $M_i = (\bigoplus_{j=1}^{i-1} P_{e_j})^{\perp}$ and n_0 is an integer such that $\lambda_{n_0} > 1$ and $\lambda_{n_0+1} \leq 1$. It is evident that $1 - \lambda_i = p(e_i, e_i)$, $i = 1, \ldots, n_0$. Then $M_- = \bigoplus_{i=1}^{n_0} P_{e_i}$ and $M_+ = M_-^{\perp}$ are negative and positive, respectively, with respect to m. Similarly as in the last theorem, we may find a Jordan decomposition for m.

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